The Logistic Model

We have seen models unrestricted growth. In exponential growth (or decay), we assume that the rate of increase (or decrease) of a population at any time t is directly proportional to the population P so that

$$\frac{dP}{dt} = kP \Rightarrow \int \frac{1}{P}\,dP = \int k\,dt \Rightarrow \ln P = kt + C$$
$$\Rightarrow P = Ce^{kt} = Pe^{rt} = A_0 e^{kt}$$

However, in many situations population growth levels off and approaches a limiting number L (the carrying capacity or M for Max) because of limited resources. In this situation the rate of increase (or decrease) is directly proportional to both (P) and $(L-P)$ This type of growth is called logistic growth

This is modeled by a differential equation which has 3 common forms, all seen in past AP exams:

1. $\frac{dP}{dt} = kP(L-P)$
2. $\frac{dP}{dt} = kP\left(1-\frac{P}{L}\right)$
3. $\frac{dy}{dt} = ky\left(1-\frac{y}{L}\right)$

equivalent:

$$\frac{dP}{dt} = kP(L-P) = kLP\left(\frac{L}{L}-\frac{P}{L}\right) = k_2 P\left(1-\frac{P}{L}\right)$$

Examples include the spread of a rumor or disease, the population growth when there are limiting resources, and even in business. Sam Walton (CEO of Wal*Mart) used the model to stop stocking items when the growth switched to concave up to concave down!

1. The he population $P(t)$ of fish in a lake satisfies the logistic differential equation

$$\frac{dP}{dy} = 3P - \frac{P^2}{6000}$$

factor to see it in std form: $= 3P\left(1-\frac{P}{18000}\right)$ OR $\frac{P}{6000}(18000-P)$

where t is measured in years, and $P(0) = 4000$.

(a) $\lim_{t\to\infty} P(t) =$ 18000

(b) What is the range of the solution curve?

$4000 \le P < 18000$ ($t=0$; $\lim_{t\to\infty}$) $[4000, 18000)$

(c) For what values of P is the solution curve increasing? Decreasing? never Justify your answer.

$\frac{dP}{dt}$: $+$ on 4000 to 18000 (If initial amt < capacity,) $P(t)$ increasing.

(d) For what values of P is the solution curve concave up? Concave down? Justify your answer.

$\frac{d^2P}{dt^2} = \left(3 - \frac{P}{3000}\right)\frac{dP}{dt} = 0, \; P = 9000$

$\frac{d^2P}{dt}$: up $+$ on 4000 to 9000; down $-$ on 9000 to 18000

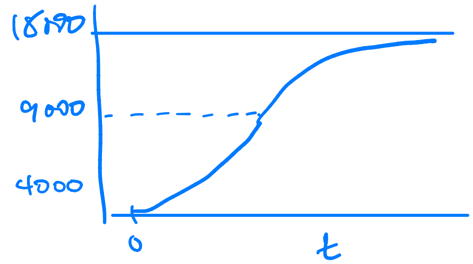
(e) Does the solution curve have an inflection point? Justify your answer.

yes at $P = 9000$, half of 18000

when $\frac{d^2P}{dt}$ changes sign

(or $\frac{dP}{dt}$ change from inc to dec)

(f) Use the information you found to sketch the graph of $P(t)$.



(g) What if the initial amount is different? What if $P(t) = 10,000$? Or if $P(t) = 20,000$? Which of your answers would change?

$P(0) = 1000$:
still inc but no P.O.I.

$P(0) = 20000$, will be decr.
until it gets to
18000

2. The function N satisfies the logistic differential equation $\dfrac{dN}{dt} = \dfrac{N}{10}\left(1 - \dfrac{N}{850}\right)$, where $N(0) = 105$. Which of the following statements are false?

if

L (or Max, or Capacity, ...)

(a) $\lim_{t\to\infty} N(t) = 850$ ✓

(b) $\dfrac{dN}{dt}$ has a maximum value when $N = \cancel{105}$ 429

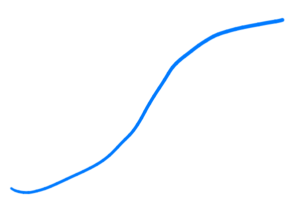
(c) $\dfrac{d^2N}{dt^2} = 0$ when $N = 425$. ✓

(d) When $N > 425$, $\dfrac{dN}{dt} > 0$ and $\dfrac{d^2N}{dt^2} < 0$ ✓

~~(e)~~ None of these

$\lim_{t\to\infty} N(t) = 850$

graph of N has P.O.I
@ $N = 425$
N is inc for $t > 0$



3. A population y changes at a rate modeled by the differential equation $\frac{dy}{dt} = 0.2y(1000 - y)$, where t is measured in years. What are all values of y for which the population is increasing at a decreasing rate?

$L = 1000$

$k = .02$

$\lim_{t\to\infty} y(t) = 1000$

$P = 500$ is P.O.I.

always increasing

$500 \leq y < 1000$

4. The number of students in a cafeteria is modeled by the function P that satisfies the logistic differential equation $\frac{dP}{dt} = \frac{1}{2000}P(200 - P)$, where t is the time in seconds and $P(0) = 25$. What is the greatest rate of change, in students per second, of the number of students in the cafeteria?

$L = 200$

$\lim_{t\to\infty} P(t) = 200$

POI @ $P = 100$

$$\left.\frac{dP}{dt}\right|_{P=100} = \frac{1}{2000}(200-100) = 5$$

greatest rate of change is 5 students per second

5. the number of moose in a national park is modeled by the function M that satisfies the logistic differential equation $\frac{dM}{dt} = 0.6M\left(1 - \frac{M}{500}\right)$, where t is the time in years and $M(0) = 50$. What is $\lim_{t\to\infty} M(t)$?

   $\hookrightarrow L$

   (a) 50
   (b) 200
   (c) 500
   (d) 1000
   (e) 2000
   (f) None of these

6. A population is modeled by the function P that satisfies the logistic differential equation $\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right)$. $\leftarrow M=12$

   (a) If $P(0) = 3$, what is $\lim_{t\to\infty} P(t)$?

   12 (Capacity or Max or limit)

   Check
   $b = \frac{12-3}{3} = 3$
   $k = \frac{1}{5}$
   Solution: $P(t) = \frac{12}{1+3e^{-t/5}} \to \frac{12}{1+0}$ as $t \to \infty$

   (b) If $P(0) = 20$, what is $\lim_{t\to\infty} P(t)$?

   $b = \frac{12-20}{20} = \frac{-8}{20} = -\frac{2}{5}$

   12 (P(0) value is not relevant)

   check
   $P(t) = \frac{12}{1-\frac{2}{5}e^{-t/5}} \to \frac{12}{1-0}$ as $t \to \infty$

   (c) If $P(0) = 3$, what value of P is the population growing the fastest?

   $\frac{M}{2} = \frac{12}{6} = 6$

   when the population is 6

   Recall
   $$\frac{dP}{dt} = \frac{P}{5} - \frac{P^2}{60}$$
   so
   $$\frac{d^2P}{dt^2} = \frac{1}{5} - \frac{2P}{60}\left(\frac{P}{5} - \frac{P^2}{60}\right) = 0$$
   $$\frac{P}{30}\left(\frac{12P - P^2}{60}\right) = \frac{1}{5}$$